Pat O'Sullivan

Mh4714 Week 2

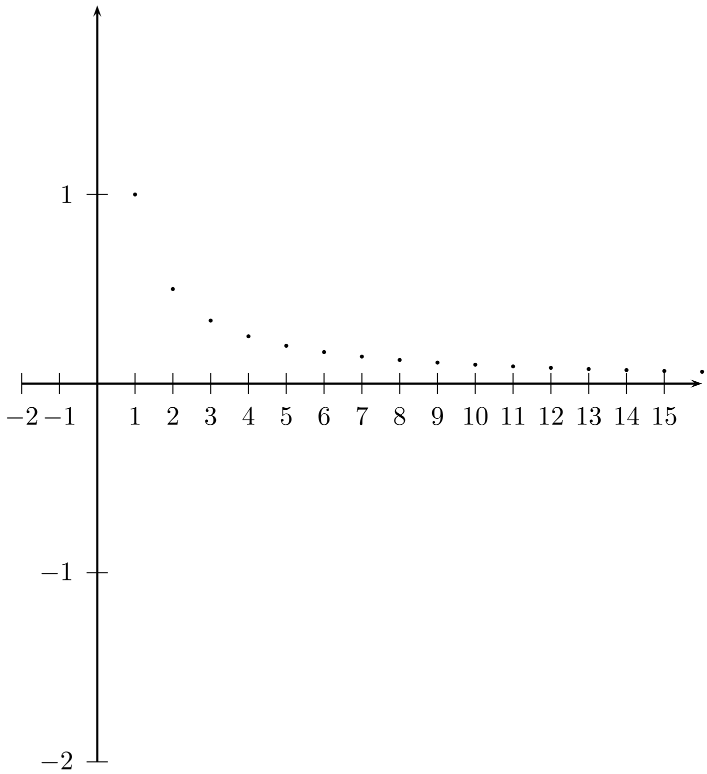
Week 2

0.0.0.1 Sequences. A sequence is a special kind of function. It is a function whose domain is the set $\mathbb{N}$.
Because they are of particular importance a particular notation has been developed.
The sequence $\{(n, f(n)) : n \in \mathbb{N}\}$ is more often referred to as:

- The sequence $\{f(n)\}_{n\in\mathbb{N}}$
- The sequence $\{f(n)\}$
- The sequence $f(1), f(2), ...f(n), ...$
- The sequence $\{a_n\}_{n\in\mathbb{N}}$ where $a_n = f(n)$.
- The sequence $\{a_n\}$ where $a_n = f(n)$.
- The sequence $a_1, a_2, ...a_n..........$ where $a_n = f(n)$.

A graph of a sequence will be "dotty" since the domain is a set of integers.

Example:
The following is a sketch of the graph of the sequence $\left\{\dfrac{1}{n}\right\}$



0.0.0.2 Series:. A *series* is a special kind of sequence which is denoted by a formal sum of the form $\sum a_n$ or $\sum_{n=1}^{\infty} a_n$ or more informally by :

$$a_1 + a_2 + a_3 + a_4 \dots$$

Each of these notations denote the sequence

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots$$

Example 0.1

- A series of the form
$$a + ar + ar^2 + ar^3 + \dots$$
is known as a *geometric series.*
- An infinite decimal is an infinite series.
For example
$$0.\dot{3} = 0.333\dots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \dots$$

This infinite decimal is a geometric series with $a = \dfrac{3}{10}, r = \dfrac{1}{10}$.

- The infinite repeating decimal $\dot{1}\dot{3}\dot{2}$ is also a geometric series since:

$$\begin{aligned}\dot{1}\dot{3}\dot{2} = 132132132\cdots &= \frac{1}{10} + \frac{3}{10^2} + \frac{2}{10^3} + \frac{1}{10^4} + \frac{3}{10^5} + \frac{2}{10^6} + \dots \\ &= \frac{132}{10^3} + \frac{132}{10^6} + \frac{132}{10^9} + \dots\end{aligned}$$

which is a geometric series with $a = \dfrac{132}{10^3}$ and $r = \dfrac{1}{10^3}$.

0.0.1 Boundedness

A set S is said to be *bounded above*, *bounded below*, *bounded*, if there is some M, m such that

$$x \le M, m \le x, m \le x \le M \quad \text{for all } x \in S.$$

A function f is said to be *bounded above*, *bounded below*, *bounded*, respectively if the range of f is bounded above, bounded below, bounded.

Example 0.2

The function x^2 is bounded below (by 0) and is not bounded above.
The function $x^2, x \in [-2, 2]$ is bounded above (by 4) and below (by 0) and is therefore bounded.
The function $\cos(x)$ is bounded.
The function x^3 is not bounded above or below.
The sequence $\left\{n^2\right\}$ is not bounded above.
The sequence $\left\{(-1)^{n-1}n^2\right\}$ is not bounded above or below.
The sequence $\{(-1)^n\}$ is bounded.

A function which is bounded above may not have a maximum value and a function which is bounded below may not have a minimum value.

Example 0.3

The function $f(x) = \begin{cases} x^2, & x \in (-2, 2) \\ 3, & x = -2 \\ 3, & x = 2 \end{cases}$ is bounded above but does not have a maximum value.

Definition 0.4

A function f is said to be *increasing* [*decreasing*] if:
$f(x_1) \leq f(x_2)$ $[f(x_1) \geq f(x_2)]$ whenever $x_1 < x_2$ for all x_1, x_2 in its domain.

A function f is said to be *strictly increasing* [*strictly decreasing*] if:
$f(x_1) < f(x_2)$ $[f(x_1) > f(x_2)]$ whenever $x_1 < x_2$ for all x_1, x_2 in its domain.

A function f which is either increasing or decreasing is said to be *monotone*.

Definition 0.5

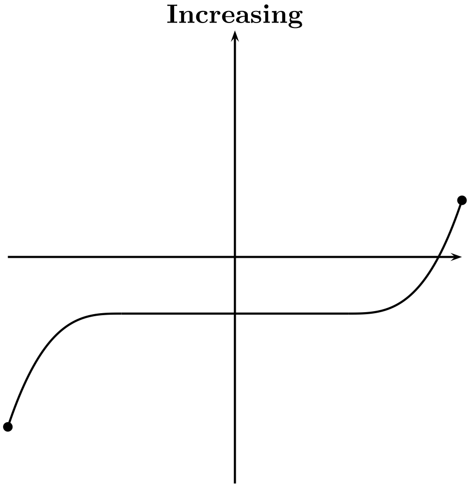
A function is said to *injective* if each x is paired with a different y.
That is, f is injective $f(x_1) = f(x_2)$ only if $x_1 = x_2$.

Example 0.6

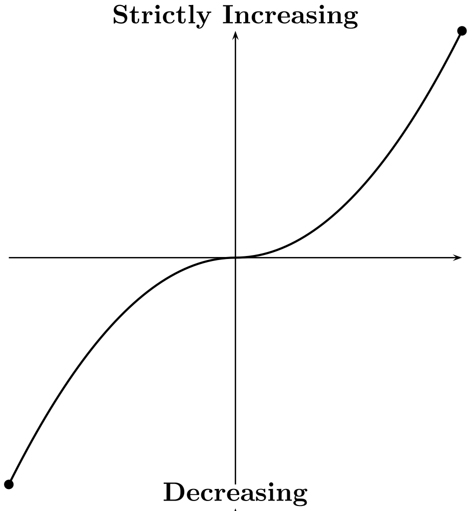
$f(x) = x^2$ is not injective because, for instance, $f(-2) = f(2)$.
$f(x) = x^2, x \in [0, \infty)$ is injective.
$f(x) = x^3$ is injective.

It is easy to prove that a function which is strictly increasing or strictly decreasing is also injective.
Sample Graphs:

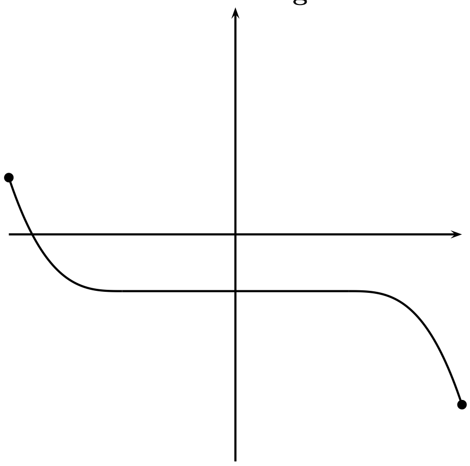
Increasing

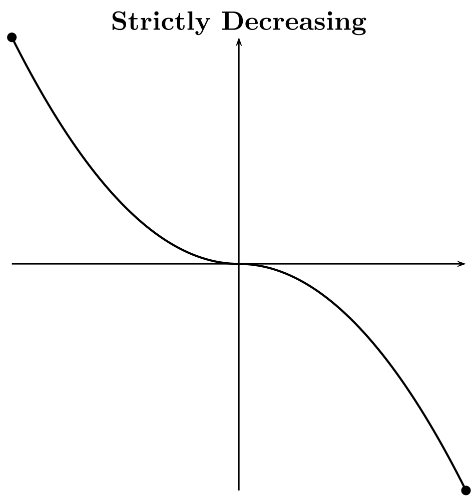


Strictly Increasing



Decreasing





0.0.2 Limits

0.0.2.1 Absolute Values.

Definition 0.7

$|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Note the following properties of absolute value:

1. $x \leq |x|$
2. $|xy| = |x||y|$.
3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.
4. $|x + y| \leq |x| + |y|$ (Triangle Inequality.)
5. $|x - y| \geq |x| - |y|$
6. $|x - y| = |y - x|$
7. $(x + y)^2 = |x + y|^2$.
8. $|x| < \epsilon \Rightarrow -\epsilon < x < \epsilon$.

9. $|x-y| < \epsilon \Rightarrow -\epsilon < x-y < \epsilon$

$$\Rightarrow -\epsilon < y-x < \epsilon$$

$$\Rightarrow Y-\epsilon < x < y+\epsilon$$

$$\Rightarrow x-\epsilon < y < x+\epsilon.$$

10. $f(x)^2 < a \Rightarrow |f(x)| < \sqrt{a}$.

Probably the most important definition in this module is that of *limit*.

Example 0.8

When we list the terms of the sequence $\left\{\dfrac{n}{n+1}\right\}$ like this

$$\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{5}{6}, \tfrac{7}{8}, \tfrac{8}{9}, \tfrac{9}{10} \cdots$$

we notice that the terms approach 1 as n increases.

It is also true that the terms "approach" 2 in the sense that they get closer to 2 also. However, the sequence clearly has a special relationship with 1 in that the terms actually become *arbitrarily* close to 1. If necessary, we can check this by looking at the values of $\left|\dfrac{n}{n+1} - 1\right|$ which measures the distance between $\dfrac{n}{n+1}$ and 1.

Terms:	$\frac{1}{2}$,	$\frac{2}{3}$,	$\frac{3}{4}$,	$\frac{5}{6}$,	$\frac{6}{7}$,	$\dots$
Distance from 1:	$\lvert\frac{1}{2}-1\rvert$,	$\lvert\frac{2}{3}-1\rvert$,	$\lvert\frac{3}{4}-1\rvert$,	$\lvert\frac{5}{6}-1\rvert$,	$\lvert\frac{6}{7}-1\rvert$	$\dots$
	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\dots$
	$\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{1}{4}$,	$\frac{1}{6}$,	$\frac{1}{7}$,	$\dots$

We see that $\left|\dfrac{n}{n+1} - 1\right|$ *becomes arbitrarily small as n gets larger.*

We make the phrase *"arbitrarily small"* precise by saying

$$\text{for any real number } \epsilon \text{ we can make } \left|\frac{n}{n+1} - 1\right| < \epsilon$$

Finally note that the phrase

$$\text{we can make } \left|\frac{n}{n+1} - 1\right| < \epsilon$$

means that we can find $N \in \mathbb{R}$ such that

$$\left|\frac{n}{n+1}-1\right|<\epsilon \text{ for all } n>N.$$

Definition 0.9

Let $\{a_n\}$ be an infinite sequence of real numbers.
$L \in \mathbb{R}$ is said to be the limit of the sequence if for each $\epsilon > 0 \in \mathbb{R}$ there is $N \in \mathbb{R}$ such that

$$|a_n - L| < \epsilon, \forall n > N.$$

Notation: We write $\lim\limits_{n\to\infty} a_n = L$

Example 0.10

1. $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Since, given $\epsilon > 0$ we have:

$$|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon \ \forall \ n > \frac{1}{\epsilon}$$

e.g. If $\epsilon = 10^{-6}$ then $\frac{1}{n} < \epsilon \ \forall \ n > 10^6$.

2. $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$. Since, given $\epsilon > 0$ we have:

$$\left|\frac{n}{n+1}-1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{1}{n+1}\right| < \epsilon \ \forall \ n > \frac{1}{\epsilon} - 1$$

A constant sequence has a limit in the most trivial way. For example, the sequence $2, 2, 2, 2, \ldots$ clearly has limit 2.

Using the curly bracket notation $\{2\}$ looks a bit odd for this sequence and so does the limit statement $\lim\limits_{n\to\infty} 2 = 2$ but we do write it from time to time.
In general, if k is a real number, the constant sequence $\{k\}$ has limit k and this is written $\lim\limits_{n\to\infty} k = k$.

Some Properties of Limits:
$\lim\limits_{n\to\infty} a_n = L_1, \lim\limits_{n\to\infty} b_n = L_2 \Rightarrow$

(i) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = L_1 + L_2$

(ii) $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n = L_1 L_2$

(iii) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L_1}{L_2}$ if $L_2 \neq 0$.

(iv) Let $\{a_n\}$ and $\{b_n\}$ be sequences and $N \in \mathbb{N}$ with $a_n \leq b_n \quad \forall n \in \mathbb{N}$.
If the sequences are convergent with $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$ then $L_1 \leq L_2$.

These properties enable us to compute limits for some compound expressions in terms of constituent parts:

Example 0.11

(i) $\lim_{n\to\infty}\frac{1}{1+nk} = \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n}+k} = \frac{\lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty}\frac{1}{n}+k}$

– using property (iii) above,

$= \frac{\lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty} k}$

– using property (i) above.

We have already proven that $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to\infty} k = k$ and so we have

$$\lim_{n\to\infty}\frac{1}{1+nh} = \frac{0}{0+h} = 0.$$

(ii) $\lim_{n\to\infty}\frac{n^2+3n}{2n^2+5} = \lim_{n\to\infty}\frac{\frac{n^2}{n^2}+\frac{3n}{n^2}}{\frac{2n^2}{n^2}+\frac{5}{n^2}} = \lim_{n\to\infty}\frac{1+\frac{3}{n}}{2+\frac{5}{n^2}} = \frac{\lim_{n\to\infty}\left(1+\frac{3}{n}\right)}{\lim_{n\to\infty}\left(2+\frac{5}{n^2}\right)}$

– using property (iii) above,

$= \frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\left(\frac{3}{n}\right)}{\lim_{n\to\infty} 2 + \lim_{n\to\infty}\left(\frac{5}{n^2}\right)}$

– using property (i) above,

$= \frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\left(\frac{3}{n}\right)}{\lim_{n\to\infty} 2 + \lim_{n\to\infty} 5 \lim_{n\to\infty}\left(\frac{1}{n}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right)}$

– using property (ii) above,

We have proven above that $\lim_{n\to\infty} 2 = 2$, $\lim_{n\to\infty} 5 = 5$ and $\lim_{n\to\infty}\frac{1}{n} = 0$ therefore we have:

$$\frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\left(\frac{3}{n}\right)}{\lim_{n\to\infty} 2 + \lim_{n\to\infty} 5 \lim_{n\to\infty}\left(\frac{1}{n}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right)} = \frac{1+0}{2+0} = \frac{1}{2}$$